RELATIVE COMMUTATOR CALCULUS IN CHEVALLEY GROUPS

ROOZBEH HAZRAT, NIKOLAI VAVILOV, AND ZUHONG ZHANG

Abstract. We revisit localisation and patching method in the setting of Chevalley groups. Introducing certain subgroups of relative elementary Chevalley groups, we develop relative versions of the conjugation calculus and the commutator calculus in Chevalley groups $G(\Phi, R)$, $\mathrm{rk}(\Phi) \geq 2$, which are both more general, and substantially easier than the ones available in the literature. For classical groups such relative commutator calculus has been recently developed by the authors in [33, 32]. As an application we prove the mixed commutator formula,

$$\big[E(\Phi, R, \mathfrak{a}), C(\Phi, R, \mathfrak{b})\big] = \big[E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})\big],$$

for two ideals $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$. This answers a problem posed in a paper by Alexei Stepanov and the second author.

بر رهگذرم هزار جا دام نهی
گویی که بگیرمت اگر گام نهی
خیام

O Life, you put many traps in my way
Dare to try, is what you clearly say
Omar Khayam

1. Introduction

One of the most powerful ideas in the study of groups of points of reductive groups over rings is localisation. It allows to reduce many important problems over arbitrary commutative rings, to similar problems for semi-local rings. Localisation comes in a number of versions. The two most familiar ones are **localisation and patching**, proposed by Daniel Quillen [54] and Andrei Suslin [64], and **localisation–completion**, proposed by Anthony Bak [7].

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Originally, the above papers addressed the case of the general linear group $\mathrm{GL}(n,R)$. Soon thereafter, Suslin himself, Vyacheslav Kopeiko, Marat Tulenbaev, Giovanni Taddei, Leonid Vaserstein, Li Fuan, Eiichi Abe, You Hong, and others proposed working versions of localisation and patching for other classical groups, such as symplectic and orthogonal ones, as well as exceptional Chevalley groups, see, for example, [34, 67, 69, 36, 37, 73] and further references in [75, 12, 61, 31]. Recently, these methods were further generalised to unitary groups, and isotropic reductive groups, by Tony Bak, Alexei Stepanov, ourselves, Victor Petrov, Anastasia Stavrova, Ravi Rao, Rabeya Basu, and others [8, 15, 9, 10, 11, 12, 16, 25, 26, 30, 49, 50, 51, 52, 55, 62]

As a matter of fact, both methods rely on a large body of common calculations, and technical facts, known as **conjugation calculus** and **commutator calculus**. Their objective is to obtain explicit estimates of the modulus of continuity in s-adic topology for conjugation by a specific matrix, in terms of the powers of s occuring in the denominators of its entries, and similar estimates for commutators of two matrices.

These calculations are *elementary*, in the strict technical sense of [86]. But being elementary, they are by no means easy. Sometimes these calculations are even called the **yoga of conjugation**, and the **yoga of commutators**, to stress the overwhelming feeling of technical strain and exertion.

A specific motivation for the present work was the desire to create tools to prove *relative* versions of structure results for Chevalley groups. Here we list three such immediate applications, in which we were particularly interested.

• Description of subnormal subgroups and subgroups normalised by the relative elementary subgroup. In full generality such description is only available for classical groups [95, 96, 97, 94], but, apart from the case of $\mathrm{GL}(n,R)$ [90, 6, 70, 38, 74, 72], sharp bounds are not obtained even in this case.

• Results on description of intemediate subgroups, such as, for example, overgroups of regularly embedded semi-simple subgroups, overgroups of exceptional Chevalley groups in an appropriate $\mathrm{GL}(n,R)$, etc., see, for example, [40, 41, 63] and [39, 76, 88] for a survey and further references.

• Generalisation of the mixed commutator formula

$$\big[E(n,R,\mathfrak{a}),\mathrm{GL}(n,R,\mathfrak{b})\big]=\big[E(n,R,\mathfrak{a}),E(n,R,\mathfrak{b})\big],$$

to exceptional Chevalley groups.

The first two problems are discussed in somewhat more detail in the last section, complete proofs are relegated to subsequent papers by the authors. Here we discuss only the third one, relative standard commutator formulae, another major objective of the present paper, apart from developing the localisation machinery itself.

The above formula was proved in the setting of general linear groups by Alexei Stepanov and the second author [87]. This formula is a common generalisation of both absolute standard commutator formulae. At the stable level, absolute commutator

formulae were first established in the foundational work of Hyman Bass [13]. In another decade, Andrei Suslin, Leonid Vaserstein, Zenon Borewicz, and the second author [64, 69, 18, 61] discovered that for commutative rings similar formulae hold for all $n \geq 3$. For two relative subgroups such formulae were proven only at the stable level, by Alec Mason [44] – [47].

However, the proof in [87] relied on a *very* strong and precise form of decomposition of unipotents [61], and was not likely to easily generalise to groups of other types. Stepanov and the second-named author raised the following problems.

• Establish the relative standard commutator formula via localisation method [87, Problem 2].

• Generalise the relative standard commutator formula to Bak's unitary groups and to Chevalley groups [87, Problem 1].

In the paper [33] the first and the third authors developed relative versions of conjugation calculus and commutator calculus in the general linear group $\mathrm{GL}(n, R)$, thus solving [87, Problem 2]. In [32] we developed a similar relative conjugation calculus in Bak's unitary groups, thus accounting for all *even* classical groups.

In the present paper, which is a direct sequel of [33, 32], we in a similar way evolve relative conjugation calculus and commutator calculus in arbitrary Chevalley groups. Actually, the present paper does not depend on the calculations from [30, 62]. Instead, here we develop *relative* versions of the yoga of conjugation, and the yoga of commutators *from scratch*, in a more general setting. The reason is that in the relative setting it is not enough to prove the continuity of conjugation by g. What we now need, is its *equi-continuity* on all congruence subgroups $G(\Phi, R, I)$. In other words, we need explicit bounds for modulus of continuity, *uniform* in the ideal I. The resulting versions of conjugation calculus and commutator calculus are both *substantially* more powerful, and *easier* than the ones available in the literature.

The overall scheme is always the same as devised by the first and the second authors in [30], and as later implemented by Alexei Stepanov and the second author [62] in a slightly more precise version, with length bounds. However, we propose several major technical innovations, and simplifications. Most importantly, following [33] and [32] we construct another base of s-adic neighbourhoods ot 1, consisting of *partially* relativised elementary groups, and prove all results not at the absolute, but at the relative level.

As an immediate application of our methods we prove the following result which, together with [32], solves [87, Problem 1] and [9, Problem 4]. Specifically, for Chevalley groups the same question was reiterated as [32, Problem 6]. Definitions of the elementary subgroup $E(\Phi, R, \mathfrak{a})$ and the full congruence subgroup $C(\Phi, R, \mathfrak{a})$ of level $\mathfrak{a} \trianglelefteq R$ are recalled in §§ 3, 4.

Theorem 1. *Let Φ be a reduced irreducible root system, $\mathrm{rk}(\Phi) \geq 2$. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume additionally that $2 \in R^*$. Further, let R be a commutative ring,*

and $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$ *be two ideals of* R*. Then*

$$[E(\Phi, R, \mathfrak{a}), C(\Phi, R, \mathfrak{b})] = [E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})].$$

Before, for exceptional groups this theorem was only known in the two special cases, when $R = \mathfrak{a}$ or when $R = \mathfrak{b}$, see [67, 71].

Condition $2 \in R^*$ in the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ is stronger than what is actually needed here, we only use it to simplify the proof of the induction base of the relative commutator calculus in § 8.

In the meantime, let us explain, why the rank 2 case, namely the types C_2 and G_2, require some serious extra care. This is due to the following circumstances.

- In these cases the elementary group $E(\Phi, R)$ is not always perfect.
- There is substantially less freedom in the Chevalley commutator formula.
- There is somewhat less freedom also in the choice of semi-simple factors.
- Most importantly, in these cases it is natural to define relative subgroups not in terms of ideals, but in terms of form ideals, or even more general structures, such as radices [20, 21].

As in [32], in the present paper we concentrate on actual calculations. History of localisation methods, philosophy behind them, and possible applications are extensively discussed in our mini-survey with Alexei Stepanov [28]. There, we also describe another remarkable recent advance, *universal* localisation developed by Stepanov [60]. For *algebraic* groups, universal localisation allows — among other things — to remove dependence on the dimension of the ground ring R in the results of [62]. Unfortunately, generalised unitary groups are not always algebraic, so that our width bounds for commutators in unitary groups [29] still depend on $\dim(\mathrm{Max}(R))$.

The paper is organised as follows. In §§ 2–4 we recall basic notation, and some background facts, used in the sequel. In § 5 we discuss injectivity of localisation homomorphism and in § 6 we calculate levels of mixed commutator subgroups. The next two sections constitute the technical core of the paper. Namely, in § 7, and in § 8 we develop relative conjugation calculus, and relative commutator calculus in Chevalley groups, respectively. After that we are in a position to give a localisation proof of Theorem 1 in § 9. On the other hand, using level calculations in § 10 we give another proof of Theorem 1, deducing it from the *absolute* standard commutator formula. There we also obtain slightly more precise results in some special situations, for instance, when $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, $\mathfrak{a} + \mathfrak{b} = R$. Finally, in § 11 we state and briefly review some further related problems.

2. Chevalley groups

As above, let Φ be a reduced irreducible root system of rank $l = \mathrm{rk}(\Phi)$, and P, $Q(\Phi) \leq P \leq P(\Phi)$ be a lattice between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. Usually, we fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$ Φ^+, Φ^- the

corresponding sets of fundamental, positive, and negative roots, respectively. Recall, that $Q(\Phi)=\mathbb{Z}\alpha_1\oplus\ldots\oplus\mathbb{Z}\alpha_l$ and $P(\Phi)=\mathbb{Z}\varpi_1\oplus\ldots\oplus\mathbb{Z}\varpi_l$, where $\varpi_!,\ldots,\varpi_l$ are the corresponding fundamental weights. Finally, $W=W(\Phi)$ denotes the Weyl group of Φ.

Further, let R be a commutative ring. We denote by $G=G_P(\Phi,R)$ the Chevalley group of type (Φ,P) over R, by $T=T_P(\Phi,R)$ a split maximal torus of G and by $E=E_P(\Phi,R)$ the corresponding (absolute) elementary subgroup. Usually P does not play role in our calculations and we suppress it in the notation.

The elementary group $E(\Phi,R)$ is generated by all root unipotents $x_\alpha(a)$, $\alpha\in\Phi$, $a\in R$, elementary with respect to T. The fact that E is normal in G means exactly that E does not depend on the choice of T.

Let G be a group. For any $x,y\in G$, ${}^xy=xyx^{-1}$ denotes the left x-conjugate of y. Let $[x,y]=xyx^{-1}y^{-1}$ denote the commutator of x and y. We will make frequent use of the following formulae,

(C1) $[x,yz]=[x,y]\cdot{}^y[x,z]$,

(C2) $[xy,z]={}^x[y,z]\cdot[x,z]$,

(C3) Hall—Witt identity

$$ {}^x\big[[x^{-1},y],z\big]={}^x\big[[y,x^{-1}]^{-1},z\big]={}^y\big[x,[y^{-1},z]\big]\cdot{}^z\big[y,[z^{-1},x]\big], $$

(C4) $[x,{}^yz]={}^y[{}^{y^{-1}}x,z]$,

(C5) $[{}^yx,z]={}^y[x,{}^{y^{-1}}z]$.

Most of the calculations in the present paper are based on the Steinberg relations

(R1) Additivity of x_α,

$$ x_\alpha(a+b)=x_\alpha(a)x_\alpha(b). $$

(R2) Chevalley commutator formula

$$ [x_\alpha(a),x_\beta(b)]=\prod_{i\alpha+j\beta\in\Phi}x_{i\alpha+j\beta}(N_{\alpha\beta ij}a^ib^j), $$

where $\alpha\neq-\beta$ and $N_{\alpha\beta ij}$ are the structure constants which do not depend on a and b. Notice, though, that for $\Phi=\mathrm{G}_2$ they may depend on the order of the roots in the product on the right hand side. The following observation was made by Chevalley himself: let $\alpha-p\beta,\ldots,\alpha-\beta,\alpha,\alpha+\beta,\ldots,\alpha+q\beta$ be the α-series of roots through β, then $N_{\alpha\beta11}=\pm(p+1)$ and $N_{\alpha\beta12}=\pm(p+1)(p+2)/2$.

Let i_Φ be the largest integer which may appear as i in a root $i\alpha+j\beta\in\Phi$ for all $\alpha,\beta\in\Phi$. Obviously $i_\Phi=1,2$ or 3, depending on whether Φ is simply laced, doubly laced or triply laced. The following result makes the proof for $\Phi\neq C_l$ slightly easier than for the symplectic case. Recall that $\mathrm{A}_1=\mathrm{C}_1$ and $\mathrm{B}_2=\mathrm{C}_2$ so that root systems of types A_1 and B_2 *are* symplectic. All roots of A_1 are long.

Our calculations in § 7 and § 8 rely on the following result, which is Lemma 2.12 in [30].

Lemma 1. *Let $\beta \in \Phi$ and either $\Phi \neq \mathrm{C}_l$ or β is short. Then there exist two roots $\gamma, \delta \in \Phi$ such that $\beta = \gamma + \delta$ and $N_{\gamma\delta 11} = 1$.*

If $\Phi = \mathrm{C}_l$, $l \geq 2$, and β is long, then there exist two roots $\gamma, \delta \in \Phi$ such that either $\beta = \gamma + 2\delta$ and $N_{\gamma\delta 12} = 1$, or $\beta = 2\gamma + \delta$ and $N_{\gamma\delta 21} = 1$.

In the sequel we also use semi-simple root elements. Namely, for $\alpha \in \Phi$ and $\varepsilon \in R^*$ we set

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \qquad h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}.$$

Let $H(\Phi, R)$ be the subgroup of $T(\Phi, R)$, generated by all $h_\alpha(\varepsilon)$, $\alpha \in \Phi$, $\varepsilon \in R^*$.

Clearly, $H(\Phi, R) \leq E(\Phi, R)$, and in fact, $H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$. In particular, for simply connected group one has

$$H_{\mathrm{sc}}(\Phi, R) = T_{\mathrm{sc}}(\Phi, R) = \mathrm{Hom}(P(\Phi), R^*).$$

For non simply connected groups, specifically, for the adjoint ones, $T(\Phi, R)$ is usually somewhat larger, than $H(\Phi, R)$. For the proof of our main theorem we have to understand, what the generators of $T(\Phi, R)$ look like in this case, see [79] for explicit constructions and many further references.

Let $\omega \in P(\Phi^\vee)$, by definition $(\alpha, \omega) \in \mathbb{Z}$ for all $\alpha \in \Phi$. Then adjoint torus contains weight elements $h_\omega(\varepsilon)$, which commute with all elements from T and satisfy the following commutator relation:

$$h_\omega(\varepsilon)x_\alpha(\xi)h_\omega(\varepsilon)^{-1} = x_\alpha(\varepsilon^{(\alpha,\o)}\xi),$$

for all $\alpha \in \Phi$ and all $\xi \in R$. For $\Phi = \mathrm{E}_8, \mathrm{F}_4$ and G_2, one has $P(\Phi) = Q(\Phi)$, in particular, in these cases $T_{\mathrm{ad}}(\Phi, R) = H_{\mathrm{ad}}(\Phi, R)$. For other cases $T_{\mathrm{ad}}(\Phi, R)$ is generated by $H_{\mathrm{ad}}(\Phi, R)$ and some weight elements.

In Section 9 we can refer to either one of the following lemmas. The first one follows from [79], Proposition 1, while the second one is well-known and obvious.

Lemma 2. *The torus $T_{\mathrm{ad}}(\Phi, R)$ is generated by $H_{\mathrm{ad}}(\Phi, R)$ and weight elements $h_\omega(\varepsilon)$, where $\varepsilon \in R^*$, and ω are the following weights*

- $\omega = \varpi_1$, *for* $\Phi = \mathrm{A}_, \mathrm{B}_l$ *and* E_6,
- $\omega = \varpi_l$, *for* $\Phi = \mathrm{C}_l$,
- $\omega = \varpi_1, \varpi_l$, *for* $\Phi = \mathrm{D}_l$,
- $\omega = \varpi_7$, *for* $\Phi = \mathrm{E}_7$.

Lemma 3. *Assume that either $\Phi \neq \mathrm{C}_l$, or $a \in \Phi$ is short. Then for any $\varepsilon \in R^*$ there exists an $h \in H(\Phi, R)$ such that $hx_\alpha(\xi)h^{-1} = x_\alpha(\varepsilon\xi)$, for all $\xi \in R$.*

In the exceptional case, where $\Phi = \mathrm{C}_l$ and $a \in \Phi$ is long, $hx_\alpha(\xi)h^{-1} = x_\alpha(\varepsilon^2\xi)$, for all $h \in H(\Phi, R)$. On the other hand, if $\alpha \in \Phi^+$ is a positive long root,

$$h_{\varpi_l}(\varepsilon)x_\alpha(\xi)h_{\varpi_l}(\varepsilon)^{-1} = x_\alpha(\varepsilon\xi).$$

Clearly, in the last case for a negative long root one has $h_{\varpi_l}(\varepsilon)x_\alpha(\xi)h_{\varpi_l}(\varepsilon)^{-1} = x_\alpha(\varepsilon^{-1}\xi)$. In the vector representation of the *extended* simply connected Chevalley group $\overline{G}(\mathrm{C}_l, R) = \mathrm{GSp}(2l, R)$ this weight element has the form

$$h_{\varpi_l}(\varepsilon) = \mathrm{diag}(\varepsilon, \dots, \varepsilon, 1, \dots, 1).$$

It follows that — with the only possible exception when $\Phi = \mathrm{C}_l$ and α is long — for any $\alpha \in \Phi$ and any $h \in T(\Phi, R)$ there exists a $g \in H(\Phi, R)$ such that $gx_\alpha(\xi)g^{-1} = hx_\alpha(\xi)h^{-1}$. In particular, $g^{-1}h$ commutes with $x_\alpha(\xi)$. However, in the exceptional case, where $\Phi = \mathrm{C}_l$ and α is long, no such g exists in general. One can only ensure the existence of such a $g \in H(\Phi, R)$ that $g^{-1}h = h_{\varpi_l}(\varepsilon)$ for some $\varepsilon \in R^*$.

3. Relative elementary subgroups

In this section we recall definitions of relative subgroups, and some basic facts used in the sequel. The usual one-parameter relative subgroups are well known. However, for multiply laced systems one should consider two-parameter relative subgroups, with one parameter corresponding to short roots, and another one to long roots. Such two-parameter relative subgroups were introduced and studied by Eiichi Abe [1]–[5] and Mike Stein [56].

Let $\mathfrak{a}$ be an additive subgroup of R. Then $E(\Phi, \mathfrak{a})$ denotes the subgroup of E generated by all elementary root unipotents $x_\alpha(t)$ where $\alpha \in \Phi$ and $t \in \mathfrak{a}$. Further, let L denote a nonnegative integer and let $E^L(\Phi, \mathfrak{a})$ denote the *subset* of $E(\Phi, \mathfrak{a})$ consisting of all products of L or fewer elementary root unipotents $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathfrak{a}$. In particular, $E^1(\Phi, \mathfrak{a})$ is the set of all $x_\alpha(t)$, $\alpha \in \Phi$, $t \in \mathfrak{a}$.

When $\mathfrak{a} \trianglelefteq R$ is an ideal of R, the elementary group $E(\Phi, \mathfrak{a})$ of level $\mathfrak{a}$ should be distinguished from the *relative* elementary subgroup $E(\Phi, R, \mathfrak{a})$ of level $\mathfrak{a}$. By definition $E(\Phi, R, \mathfrak{a})$ is the normal closure of $E(\Phi, \mathfrak{a})$ in the absolute elementary subgroup $E(\Phi, R)$. In general $E(\Phi, R, \mathfrak{a})$ is *not* generated by elementary transvections of level $\mathfrak{a}$, below we describe its generators for $\mathrm{rk}(\Phi) \geq 2$. The following result can be found in [56, 68].

Lemma 4. *In the case $\Phi \neq \mathrm{C}_l$ one has $E(\Phi, \mathfrak{a}) \geq E(\Phi, R, \mathfrak{a}^2)$. In the exceptional case $\Phi = \mathrm{C}_l$ one has $E(\Phi, \mathfrak{a}) \geq E(\Phi, R, (2R + \mathfrak{a})\mathfrak{a}^2)$.*

Let $\mathfrak{a}$ be an ideal of R. Denote by $\mathfrak{a}_2$ the ideal, generated by 2ξ and ξ^2 for all $\xi \in \mathfrak{a}$. The first component $\mathfrak{a}$ of an admissible pair $(\mathfrak{a}, \mathfrak{b})$ is an ideal of R, parametrising short roots. When $\Phi \neq \mathrm{C}_l$ the second component $\mathfrak{b}$, $\mathfrak{a}_2 \leq \mathfrak{b} \leq \mathfrak{a}$, is also an ideal, parametrising long roots. In the exceptional case $\Phi = \mathrm{C}_l$ the second component $\mathfrak{b}$ is an additive subgroup stable under multiplication by ξ^2, $\xi \in R$ (in other words, it is a relative form parameter in the sense of Bak [12, 24, 31]). A similar notion can be introduced for the type G_2 as well, but in this case one should replace 2 by 3 *everywhere* in the above definition.

Now the *relative* elementary subgroup, corresponding to an admissible pair $(\mathfrak{a}, \mathfrak{b})$, is defined as follows:

$$E(\Phi, R, \mathfrak{a}, \mathfrak{b}) = \langle x_\alpha(\xi), \alpha \in \Phi_s, \xi \in \mathfrak{a};\ x_\beta(\zeta), \beta \in \Phi_l, \zeta \in \mathfrak{b} \rangle^{E(\Phi,R)}.$$

where Φ_s and Φ_l are the sets of long and short roots in Φ, respectively. The following results can be found in [56, 3, 4].

Lemma 5. *Let* $\mathrm{rk}(\Phi) \geq 2$. *When* $\Phi = \mathrm{B}_2$ *or* $\Phi = \mathrm{G}_2$ *assume moreover that* R *has no residue fields* $\mathbb{F}_2$ *of* 2 *elements. Then the elementary subgroup* $E(\Phi, R, \mathfrak{a}, \mathfrak{b})$ *is* $E(\Phi, R)$*-perfect, in other words,*

$$\big[E(\Phi, R), E(\Phi, R, \mathfrak{a}, \mathfrak{b})\big] = E(\Phi, R, \mathfrak{a}, \mathfrak{b}).$$

In particular, $E(\Phi, R)$ *is perfect.*

Lemma 6. *As a subgroup* $E(\Phi, R, \mathfrak{a}, \mathfrak{b})$ *is generated by the elements*

$$z_\alpha(\xi, \zeta) = x_{-\alpha}(\zeta) x_\alpha(\xi) x_{-\alpha}(-\zeta),$$

where $\xi \in \mathfrak{a}$ *for* $\alpha \in \Phi_s$ *and* $\xi \in \mathfrak{b}$ *for* $\alpha \in \Phi_l$, *while* $\zeta \in R$.

Actually, in the sequel we mostly use these results in the special case, where $\mathfrak{a} = \mathfrak{b}$.

4. Congruence subgroups

Usually, one defines congruence subgroups as follows. An ideal $\mathfrak{a} \trianglelefteq R$ determines the reduction homomorphism $\rho_\mathfrak{a} : R \longrightarrow R/\mathfrak{a}$. Since $G(\Phi, \_)$ is a functor from rings to groups, this homomorphism induces reduction homomorphism $\rho_\mathfrak{a} : G(\Phi, R) \longrightarrow G(\Phi, R/\mathfrak{a})$.

• The kernel of the reduction homomorphism $\rho_\mathfrak{a}$ modulo $\mathfrak{a}$ is called the principal congruence subgroup of level $\mathfrak{a}$ and is denoted by $G(\Phi, R, \mathfrak{a})$.

• The full pre-image of the centre of $G(\Phi, R/\mathfrak{a})$ with respect to the reduction homomorphism $\rho_\mathfrak{a}$ modulo $\mathfrak{a}$ is called the full congruence subgroup of level $\mathfrak{a}$, and is denoted by $C(\Phi, R, \mathfrak{a})$.

A more general notion of congruence subgroup was introduced in [27]. Namely, consider a linear action of G on a right R-module V and let $U \leq V$ be a G-submodule. Then we can define a set

$$G(V, U) = \big\{ g \in G \mid \forall v \in V,\ gv - v \in U \big\}.$$

This set is in fact a *normal* subgroup of G.

An application of this construction to a Chevalley group $G = G(\Phi, R)$ and its rational module V allows us to recover the usual subgroups. For any module V and any ideal $\mathfrak{a} \trianglelefteq R$ the product $U = V\mathfrak{a}$ is a G-submodule. The following result is [27, Lemma 6].

Lemma 7. *When* V *is a faithful rational module,* $G(V, V\mathfrak{a}) = G(\Phi, R, \mathfrak{a})$ *is the usual principal congruence subgroup of level* $\mathfrak{a}$.

In matrix language, this lemma means that the principal congruence subgroup of level $\mathfrak{a}$ can be defined as

$$G(\Phi, R, \mathfrak{a}) = G(\Phi, R) \cap \mathrm{GL}(n, R, \mathfrak{a}),$$

for any *faithful* rational representation $G(\Phi, R) \leq \mathrm{GL}(n, R)$.

Clearly, for *any* rational representation $\phi : G(\Phi, R) \longrightarrow \mathrm{GL}(n, R)$, one has inclusions

$$\phi^{-1}\big(G(\Phi, R) \cap \mathrm{GL}(n, R, \mathfrak{a})\big) \leq C(\Phi, R, \mathfrak{a}) \leq \phi^{-1}\big(G(\Phi, R) \cap C(n, R, \mathfrak{a})\big),$$

for the full congruence subgroup. In the general case there is no reason, why either of these inclusions should be an equality. However, there is one important special case, where the left inclusion becomes an equality [27, Lemma 7].

Lemma 8. *When $V = L$ is the Lie algebra of $G(\Phi, R)$, considered as the adjoint module, then $G(L, L\mathfrak{a}) = C(\Phi, R, \mathfrak{a})$ is the usual full congruence subgroup of level $\mathfrak{a}$.*

The following result, Theorem 2 of [27], asserts that three possible definitions of the full congruence subgroup coincide.

Lemma 9. *Let Φ be a reduced irreducible root system of rank ≥ 2, R be a commutative ring, $(\mathfrak{a}, \mathfrak{b})$ an admissible pair. Then the following four subgroups coincide:*

$$\begin{aligned} C(\Phi, R, \mathfrak{a}, \mathfrak{b}) &= \big\{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq E(\Phi, R, \mathfrak{a}, \mathfrak{b})\big\} \\ &= \big\{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq C(\Phi, R, \mathfrak{a}, \mathfrak{b})\big\} \\ &= \big\{g \in G(\Phi, R) \mid [g, G(\Phi, R)] \leq C(\Phi, R, \mathfrak{a}, \mathfrak{b})\big\}. \end{aligned}$$

In fact, in [27] we established standard commutator formulae for the case, where one argument is an *absolute* subgroup, whereas the second argument is a relative subgroup with *two* parameters. In particular, the following result is Theorem 1 of [27]. Of course, in all cases, except Chevalley groups of type F_4, it was known before, [12, 50, 21].

Lemma 10. *Let Φ be a reduced irreducible root system of rank ≥ 2, R be a commutative ring, $(\mathfrak{a}, \mathfrak{b})$ an admissible pair. In the case, where $\Phi = \mathrm{B}_2$ or $\Phi = \mathrm{G}_2$ assume moreover that R has no residue fields $\mathbb{F}_2$ of 2 elements. Then the following standard commutator formulae holds*

$$\big[G(\Phi, R), E(\Phi, R, \mathfrak{a}, \mathfrak{b})\big] = \big[E(\Phi, R), C(\Phi, R, \mathfrak{a}, \mathfrak{b})\big] = E(\Phi, R, \mathfrak{a}, \mathfrak{b}).$$

We will use the following form of Gauß decomposition, stated by Eiichi Abe [1, 5]. Namely, let $\mathfrak{a} \trianglelefteq R$ be an ideal of R. We denote by $T(\Phi, R, \mathfrak{a})$ the subbgroup of the split maximal torus $T(\Phi, R)$, consisting of all elements congruent to e modulo $\mathfrak{a}$,

$$T(\Phi, R, \mathfrak{a}) = T(\Phi, R) \cap G(\Phi, R, \mathfrak{a}).$$

As usual, we set

$$U(\Phi, \mathfrak{a}) = \big\langle x_\alpha,\ \alpha \in \Phi^+,\ a \in \mathfrak{a}\big\rangle, \qquad U^-(\Phi, \mathfrak{a}) = \big\langle x_\alpha,\ \alpha \in \Phi^-,\ a \in \mathfrak{a}\big\rangle.$$

Obviously, $U(\Phi, \mathfrak{a}), U^-(\Phi, \mathfrak{a}) \leq E(\Phi, \mathfrak{a})$.

Lemma 11. *Let $\mathfrak{a}$ be an ideal of R contained in the Jacobson radical* $\mathrm{Rad}(R)$*. Then*
$$G(\Phi, R, \mathfrak{a}) = U(\Phi, \mathfrak{a})T(\Phi, R, \mathfrak{a})U^-(\Phi, \mathfrak{a}).$$

We will mostly use this lemma in the following form, see [30], Lemma 2.10.

Lemma 12. *If $\mathfrak{a}$ is an ideal of local ring R then*
$$G(\Phi, R, \mathfrak{a}) = E(\Phi, \mathfrak{a})T(\Phi, R, \mathfrak{a}).$$

5. Injectivity of localisation homomorphism

Let us fix some notation. Let R be a commutative ring with 1, S be a multiplicative system in R and $S^{-1}R$ be the corresponding localisation. We will mostly use localisation with respect to the two following types of multiplicative systems.

- If $s \in R$ and the multiplicative system S coincides with $\langle s\rangle = \{1, s, s^2, \dots\}$ we usually write $\langle s\rangle^{-1}R = R_s$.

- If $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal in R, and $S = R \setminus \mathfrak{m}$, we usually write $(R \setminus \mathfrak{m})^{-1}R = R_{\mathfrak{m}}$.

We denote by $F_S : R \longrightarrow S^{-1}R$ the canonical ring homomorphism called the localisation homomorphism. For the two special cases mentioned above, we write $F_s : R \longrightarrow R_s$ and $F_{\mathfrak{a}} : R \longrightarrow R_{\mathfrak{a}}$, respectively.

When we write an element as a fraction, like a/s or $\dfrac{a}{s}$, we *always* think of it as an element of some localisation $S^{-1}R$, where $s \in S$. If s were actually invertible in R, we would have written as^{-1} instead.

5.1. The property of these functors which will be crucial for what follows is that they *commute with direct limits*. In other words, if $R = \varinjlim R_i$, where $\{R_i\}_{i\in I}$ is an inductive system of rings, then $X(\Phi, \varinjlim R_i) = \varinjlim X(\Phi, R_i)$. We will use this property in the following two situations.

- First, let R_i be the inductive system of all finitely generated subrings of R with respect to inclusion. Then $X = \varinjlim X(\Phi, R_i)$, which reduces most of the proofs to the case of Noetherian rings.

- Second, let S be a multiplicative system in R and R_s, $s \in S$, the inductive system with respect to the localisation homomorphisms: $F_t : R_s \longrightarrow R_{st}$. Then $X(\Phi, S^{-1}R) = \varinjlim X(\Phi, R_s)$, which allows to reduce localisation in any multiplicative system to principal localisations.

Our proofs rely on the injectivity of localisation homomorphism F_s. On the group $G(\Phi, R)$ itself it is seldom injective, but its restrictions to appropriate congruence subgroups often are, see the discussion in [28]. Below we cite two important typical cases, Noetherian rings [7] and semi-simple rings [62].

Lemma 13. *Suppose R is Noetherian and $s \in R$. Then there exists a natural number k such that the homomorphism $F_s : G(\Phi, R, s^kR) \longrightarrow G(\Phi, R_s)$ is injective.*

Proof. The homomorphism $F_s : G(\Phi, R, s^k R) \longrightarrow G(\Phi, R_s)$ is injective whenever $F_s : s^k R \longrightarrow R_s$ is injective. Let $\mathfrak{a}_i = \mathrm{Ann}_R(s^i)$ be the annihilator of s^i in R. Since R is Noetherian, there exists k such that $\mathfrak{a}_k = \mathfrak{a}_{k+1} = \dots$. If $s^k a$ vanishes in R_s, then $s^i s^k a = 0$ for some i. But since $\mathfrak{a}_{k+i} = \mathfrak{a}_k$, already $s^k a = 0$ and thus s^R injects in R_s. □

Lemma 14. *If* $\mathrm{Rad}(R) = 0$*, then* $F_s : G(\Phi, R, sR) \longrightarrow G(\Phi, R_s)$ *is injective for all* $s \in R$*,* $s \neq 0$*.*

Proof. It suffices to prove that $F_s : sR \longrightarrow R_s$ is injective. Suppose that $s\xi \in sR$ goes to 0 in R_s. Then there exists an $m \in \mathbb{N}$ such that $s^m s\xi = 0$. It follows that $(s\xi)^{m+1} = 0$ and since R is semi-simple, $s\xi = 0$. □

In [30] we used reduction to Noetherian rings, whereas in [62] reduction to semi-simple rings was used instead.

6. Levels of mixed commutators

In this section we establish some obvious facts, concerning the lower and the upper levels of mixed commutators

$$\big[E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})\big] \leq \big[G(\Phi, R, \mathfrak{a}), G(\Phi, R, \mathfrak{b})\big].$$

Unlike most other results of the present paper, the next lemma also holds for $\mathrm{rk}(\Phi) = 1$.

Lemma 15. *Then for any two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of the ring* R *one has*

$$E(\Phi, R, \mathfrak{a}, \mathfrak{c}) E(\Phi, R, \mathfrak{b}, \mathfrak{d}) = E(\Phi, R, \mathfrak{a} + \mathfrak{b}, \mathfrak{c} + \mathfrak{d}).$$

Proof. Additivity of the elementary root unipotents $x_\alpha(a + b) = x_\alpha(a) x_\alpha(b)$, where $\alpha \in \Phi$, while $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ for $\alpha \in \Phi_s$, and $a \in \mathfrak{c}$, $b \in \mathfrak{d}$ for $\alpha \in \Phi_l$, implies that the left hand side contains *generators* of the right hand side. The product of two normal subgroups is normal in $E(\Phi, R)$. □

As a preparation to the calculation of lower level, we generalise Lemma 4. It is a toy version of the main results of the present paper, whose proof heavily depends on Lemma 6. There one considers

$$z_\alpha(ab, \zeta) = {}^{x_{-\alpha}(\zeta)} x_\alpha(ab).$$

The idea is to express $x_\alpha(ab)$ as the commutator of two root elements $[x_\beta(a), x_\gamma(b)]$, where $\beta + \gamma = \alpha$, plus, possibly, some tail. Now, neither the roots β, γ, nor the roots appearing in the tail, are opposite to $-\alpha$, and thus we can distribute conjugation by $x_{-\alpha}(\zeta)$ and apply the Chevalley commutator formula to each occuring factor. The first explicit appearance of this idea, which we were able to trace in the literature, was in Bass—Milnor—Serre foundational work [14].

Lemma 16. *Let* $\mathrm{rk}(\Phi) \geq 2$ *and further let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* R. *Assume that either* $\Phi \neq \mathrm{C}_l$, *or* $2 \in R^*$. *Then one has*

$$E(\Phi, R, \mathfrak{ab}) \leq E(\Phi, \mathfrak{a} + \mathfrak{b}).$$

In the exceptional case, where $\Phi = \mathrm{C}_l$ *and* $2 \notin R^*$ *one has*

$$E(\Phi, R, \mathfrak{ab}, \mathfrak{ab}^2 + 2\mathfrak{ab} + \mathfrak{a}^2\mathfrak{b}) \leq E(\Phi, \mathfrak{a} + \mathfrak{b}).$$

Proof. By Lemma 6 is suffices to find conditions on ξ which imply that $z_\alpha(\xi, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$ for each root $\alpha \in \Phi$ and $\zeta \in R$.

General case. First, assume that α is short or $\Phi \neq \mathrm{C}_l$. By Lemma 1 there exist roots β and γ such that $\beta + \gamma = \alpha$ and $N_{\beta\gamma 11} = 1$. In this case we prove that $z_\alpha(ab, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$ for each root $\alpha \in \Phi$ and all $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $\zeta \in R$. With this end we decompose $x_\alpha(ab)$ as follows:

$$x_\alpha(ab) = [x_\beta(a), x_\gamma(b)] \prod x_{i\beta + j\gamma}(-N_{\gamma\delta ij} a^i b^j),$$

where the product on the right hand side is taken over all roots $i\beta + j\gamma \neq \alpha$. Conjugating this equality by $x_{-\alpha}(\zeta)$, we obtain an expression of $z_\alpha(ab, \zeta)$ as a product of elementary root unipotents belonging either to $E(\Phi, \mathfrak{a})$ or to $E(\Phi, \mathfrak{b})$, or, as in the case of factors occuring in the tail, to both.

Case $\Phi = \mathrm{C}_l$**.** This leaves us with the analysis of the exceptional case, where $\Phi = \mathrm{C}_l$ and the root α is long. We will have to use several instances of the Chevalley commutator formula.

First of all, there exist *short* roots β and γ such that $\beta + \gamma = \alpha$ and $N_{\beta\gamma 11} = 2$. Thus,

$$x_\alpha(2ab) = [x_\beta(a), x_\gamma(b)],$$

for all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Now, exactly the same argument, as in the general case, shows that $z_\alpha(2ab, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$. This shows that when $2 \in R^*$ we again recover the general answer.

By Lemma 1 there exist a long root root β and a short root γ such that $\beta + 2\delta = \alpha$ and $N_{\beta\gamma 12} = \pm 1$. Without loss of generality we can assume that $N_{\beta\gamma 12} = \pm 1$, otherwise we would just replace the $x_\gamma(a)$ in the following formula by $x_\gamma(-a)$. We decompose $x_\beta(s^h t^m a)$ as follows:

$$x_\beta(ab^2) = [x_\gamma(a), x_\delta(b)] x_{\gamma+\delta}(-N_{\gamma\delta 11} ab),$$

or all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Now, exactly the same argument, as in the general case, shows that $z_\alpha(ab^2, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$.

Interchanging a and b in the above formula, we see, that $z_\alpha(ab^2, \zeta) \in E(\Phi, \mathfrak{a} + \mathfrak{b})$. To finish the proof, it remains only to refer to the preceding lemma. $\square$

In the next lemma we calculate the *lower* level of the mixed commutator subgroup.

Lemma 17. *Let* $\mathrm{rk}(\Phi) \geq 2$. *Then for any two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of the ring* R *one has the following inclusion*

$$E(\Phi, R, \mathfrak{a}\mathfrak{b}) \leq \big[E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})\big].$$

Proof. The mixed commutator of two normal subgroups is normal. Thus, it suffices to prove that

$$E(\Phi, \mathfrak{a}\mathfrak{b}) \leq \big[E(\Phi, R, \mathfrak{a}), E(\Phi, R, \mathfrak{b})\big],$$

and the result will automatically follow.

Let $\alpha \in \Phi$. First, assume that α can be embedded in a root system of type A_2. Then there exist roots $\beta, \gamma \in \Phi$, of the same length as α such that $\alpha = \beta + \gamma$, and $N_{\beta\gamma 11} = 1$. Then

$$[x_\beta(a), x_\gamma(b)] = x_\alpha(ab).$$

This proves the lemma for simply laced Chevalley groups, and for the Chevalley group of type F_4. It also proves necessary inclusions for *short* roots in Chevalley groups of type C_l, $l \geq 3$, and for *long* roots in Chevalley groups of type B_l, $l \geq 3$, and of type G_2. □

Not to overburden the present paper with technical details, here we only consider the usual relative subgroups depending on one parameter. To illustrate, why we do this, let us state a general version of Lemma 13, with form parameters.

Lemma 18. *Let* $\mathrm{rk}(\Phi) \geq 2$. *Then for any two for ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of the ring* R *one has the following inclusions*

$$E(\Phi, R, \mathfrak{a}\mathfrak{b}, i_\Phi \mathfrak{a}\mathfrak{b} + \mathfrak{a}^2\mathfrak{c} + \mathfrak{b}^2\mathfrak{d}) \leq \big[E(\Phi, R, \mathfrak{a}, \mathfrak{c}), E(\Phi, R, \mathfrak{b}, \mathfrak{d})\big].$$

Next lemma bounds the upper level of mixed commutator subgroups. Observe, that it also holds for $\mathrm{rk}(\Phi) = 1$.

Lemma 19. *Let* $\mathrm{rk}(\Phi) \geq 1$. *Then for any two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of the ring* R *one has the following inclusion*

$$\big[G(\Phi, R, \mathfrak{a}), C(\Phi, R, \mathfrak{b})\big] \leq G(\Phi, R, \mathfrak{a}\mathfrak{b}).$$

Proof. Consider a faithful rational representation $G(\Phi, R) \leq \mathrm{GL}(n, R)$. Then

$$G(\Phi, R, \mathfrak{a}) \leq \mathrm{GL}(n, R, \mathfrak{a}), \qquad C(\Phi, R, \mathfrak{b}) \leq C(n, R, \mathfrak{b}).$$

Now, by lemma 5 of [89] one has

$$\big[G(\Phi, R, \mathfrak{a}), C(\Phi, R, \mathfrak{b})\big] \leq \big[\,\mathrm{GL}(n, R, \mathfrak{a}), C(n, R, \mathfrak{b})\big] \leq \mathrm{GL}(n, R, \mathfrak{a}\mathfrak{b}).$$

Since the left hand side is a subgroup of $G(\Phi, R)$, by lemma 7 we get

$$\big[G(\Phi, R, \mathfrak{a}), C(\Phi, R, \mathfrak{b})\big] \leq G(\Phi, R) \cap \mathrm{GL}(n, R, \mathfrak{a}\mathfrak{b}) = G(\Phi, R, \mathfrak{a}\mathfrak{b}).$$

□

7. Relative conjugation calculus

This section, and the next one constitute the technical core of the paper. Here, we develop a relative version of the conjugation calculus in Chevalley groups, whereas in the next section we evolve a relative version of the commutator calculus. Throughout this section we assume $\mathrm{rk}(\Phi) \geq 2$.

For future applications we allow two localisation parameters. With this end we fix *two* elements $s, t \in R$ and look at the localisation

$$R_{st} = (R_s)_t = (R_t)_s.$$

All calculations in this and the next sections take place in $E(\Phi, R_{st})$. Thus, when we write something like $E(\Phi, s^p t^q R)$, or $x_\alpha(s^p a)$, what we *really* mean, is $E\big(\Phi, F_s(s^p t^q R)\big)$, or $x_\alpha(F_s(s^p a))$, respectively, but we suppress F_s in our notation. This shouldn't lead to a confusion, since here we *never* refer to elements or subgroups of $G(\Phi, R)$.

The overall strategy in this and the next sections is *exactly* the same, as in the proofs of Lemmas 3.1 and 4.1 of [30] and and in the proofs of Lemmas 8–10 of [62].

However, now we wish to do it at the relative, rather than absolute level. In other words, we have to introduce another parameter belonging to an ideal $\mathfrak{a} \trianglelefteq R$. The difference with the existing versions of localisation is that whereas powers of localising elements s and t are at our disposal, and can be distributed among the factors, the ideal $\mathfrak{a}$ is fixed, and cannot be distributed.

The first main objective of the conjugation calculus is to establish that conjugation by a fixed matrix $g \in G(\Phi, R_s)$ is continuous in s-adic topology. In the proof one uses a base of neighborhoods of e and establishes that for any such neighborhood V there exists another neighborhood U such that ${}^gU \subseteq V$. Usually, one takes either elementary subgroups $E(\Phi, s^k\mathfrak{a})$ of level $s^k\mathfrak{a}$, or *relative* elementary subgroups $E(\Phi, R, s^k\mathfrak{a})$ of level $s^k\mathfrak{a}$, as a base.

However, both choices are not fully satisfactory in that they lead to extremelly onerous calculations. The reason is that the first of these choices is too small as the neighbourhood on the right hand side, while the second of these choices is too large as the neighbourhood on the left hand side. The solution proposed for $\mathrm{GL}(n, R)$ in [33] and later applied to unitary groups in [32] consists in selecting another base of neightborhoods

$$E(\Phi, s^k\mathfrak{a}) \leq E(\Phi, s^kR, s^k\mathfrak{a}) \leq E(\Phi, R, s^k\mathfrak{a}),$$

which is much better balanced with respect to conjugation. The following definition embodies the gist of this method.

Definition 7.1. Let R be a commutative ring, $\mathfrak{a}$ an ideal of R and $s \in R$. For a positive integer k, define

$$E(\Phi, s^kR, s^k\mathfrak{a}) = E(\Phi, s^k\mathfrak{a})^{E(\Phi, s^kR)}$$

as the normal closure of $E(\Phi, s^k\mathfrak{a})$ in $E(\Phi, s^kR)$, i.e., the group generated by ${}^e x_\alpha(s^k a)$ where $e \in E^K(\Phi, s^kR)$, for some positive integer K, $a \in \mathfrak{a}$ and $\alpha \in \Phi$.

The following lemma is a relative version of Lemma 3.1 of [30] and of Lemma 8 of [62]. Observe, that we could not simply put $E(\Phi, s^p t^q \mathfrak{a})$ on the right hand side. While the powers of s and t can be distributed among the factors on the right hand side in the calculations below, this is not the case for the parameter $a \in \mathfrak{a}$. This is why we need conjugates by elements of $E(\Phi, s^p t^q R)$.

Observe, that the proof works in terms of roots alone, and thus one gets *uniform* estimates for the powers of s and t, which do not depend on the ideal $\mathfrak{a}$. This circumstance, the *equi-continuity* of conjugation by $g \in G(\Phi, R_s)$ on congruence subgroups, is extremely important, and will be repeatedly used in the sequel.

Lemma 20. *If p, q and k are given, there exist h and m such that*

$$^{E^1\left(\Phi, \frac{1}{s^k}R\right)}E(\Phi, s^h t^m \mathfrak{a}) \subseteq E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}).$$

Such h and m depend on Φ, k, p and q alone, but does not depend on the ideal $\mathfrak{a}$.

Proof. Since by definition $E(\Phi, s^h t^m \mathfrak{a})$ is generated by $x_\beta(s^h t^m a)$, where $\beta \in \Phi$ and $a \in \mathfrak{a}$, it suffices to show that there exist h and m such that

$$^{x_\alpha\left(\frac{r}{s^k}\right)}x_\beta(s^h t^m a) \in E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}),$$

for any $x_\alpha(r/s^k) \in E^1\big(\Phi, \frac{1}{s^k}R\big)$ and any $x_\beta(s^h t^m a) \in E(\Phi, s^h t^m \mathfrak{a})$.

Case 1. Let $\alpha \neq -\beta$ and set $h \geq i_\Phi k + p + 1$, $m \geq q$. By the Chevalley commutator formula,

$$x_\alpha\bigg(\frac{r}{s^k}\bigg)x_\beta(s^h t^m a)x_\alpha\bigg(-\frac{r}{s^k}\bigg) = \prod_{i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}\Bigg(N_{\alpha\beta ij}\bigg(\frac{r}{s^k}\bigg)^i (s^h t^m a)^j\Bigg)x_\beta(s^h t^m a)$$

and a quick inspection shows that the right hand side of the above equality is in $E^L(\Phi, s^p t^q \mathfrak{a})$, where $L = 2, 3$ or 5, depending on whether Φ is simply laced, doubly laced or triply laced. Clearly,

$$E^L(\Phi, s^p t^q \mathfrak{a}) \subseteq E(\Phi, s^p t^q \mathfrak{a}) \leq E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}).$$

Case 2. Let $\alpha = -\beta$ and one of the following holds: β is short or $\Phi \neq \mathrm{C}_l$. By Lemma 1 there exist roots γ and δ such that $\gamma + \delta = \beta$ and $N_{\gamma\delta 11} = 1$. We set $h = 2(i_\Phi k + p + 1)$, $m = 2q$, and decompose $x_\beta(s^h t^m a)$ as follows:

$$x_\beta(s^h t^m a) = [x_\gamma(s^{h/2} t^{m/2}), x_\delta(s^{h/2} t^{m/2} b)] \prod x_{i\gamma+j\delta}(-N_{\gamma\delta ij}(s^{h/2} t^{m/2})^i (s^{h/2} t^{m/2} a)^j),$$

where the product on the right hand side is taken over all roots $i\gamma + j\delta \neq \beta$.

Conjugating this expression by $x_\alpha\big(\frac{r}{s^k}\big)$ we get

$$^{x_\alpha\left(\frac{r}{s^k}\right)}x_\beta(s^h t^m a) = \Big[{}^{x_\alpha\left(\frac{r}{s^k}\right)}x_\gamma(s^{h/2} t^{m/2}), {}^{x_\alpha\left(\frac{r}{s^k}\right)}x_\delta(s^{h/2} t^{m/2} a)\Big]\cdot$$
$$\prod {}^{x_\alpha\left(\frac{r}{s^k}\right)}x_{i\gamma+j\delta}\Big(-N_{\gamma\delta ij}(s^{h/2} t^{m/2})^i (s^{h/2} t^{m/2} a)^j\Big).$$

Obviously, γ, δ and all the roots $i\gamma + j\delta \neq \beta$, occuring in the product, are distinct from $-\alpha$. Now, by Case 1 the first element of the commutator belongs to $E(\Phi, s^p t^q R)$,

while the second element of the commutator, and all factors of the product belong to $E(\Phi, s^p t^q \mathfrak{a})$. Since $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$ is normalised by $E(\Phi, s^p t^q R)$, it follows that each term on right hand side sits in $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$.

Case 3. Let $\Phi = \mathrm{C}_l$ and $\alpha = -\beta$ be a long root. By Lemma 1 there exist roots γ and δ such that either $\gamma + 2\delta = \beta$ and $N_{\gamma\delta 12} = 1$, or $2\gamma + \delta = \beta$ and $N_{\gamma\delta 21} = 1$. We look at the first case, the second case is similar. Alternatively, if $N_{\gamma\delta 12} = -1$, one could change the sign of $x_\gamma(a)$ in the following formula by $x_\gamma(-a)$. We set $h = 3(i_\Phi k + p + 1)$ and $m = 3q$, and decompose $x_\beta(s^h t^m a)$ as follows:

$$x_\beta(s^h t^m a) = \big[x_\gamma(s^{h/3} t^{m/3} a), x_\delta(s^{h/3} t^{m/3})\big] \cdot x_{\gamma+\delta}(-N_{\gamma\delta 11} s^{2h/3} t^{2m/3} a),$$

Conjugating this expression by $x_\alpha\big(\frac{r}{s^k}\big)$ we get

$$\begin{aligned} {}^{x_\alpha\left(\frac{r}{s^k}\right)} x_\beta(s^h t^m a) = \Big[{}^{x_\alpha\left(\frac{r}{s^k}\right)} x_\gamma(s^{h/3} t^{m/3} a), {}^{x_\alpha\left(\frac{r}{s^k}\right)} x_\delta(s^{h/3} t^{m/3})\Big] \cdot \\ {}^{x_\alpha\left(\frac{r}{s^k}\right)} x_{\gamma+\delta}(-N_{\gamma\delta 11} s^{2h/3} t^{2m/3} a). \end{aligned}$$

As in Case 2, we can apply Case 1 to each conjugate on the right hand side. The first element of the commutator, and the last factor belong to $E(\Phi, s^p t^q \mathfrak{a})$, while the second element of the commutator belongs to $E(\Phi, s^p t^q R)$. Again, it remains only to recall that $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$ is normalised by $E(\Phi, s^p t^q R)$. $\square$

Now, the trick is that the elementary group $E(\Phi, s^h t^m \mathfrak{a})$ on the left hand side can be *effortlessly* replaced by $E(\Phi, s^h t^m R, s^h t^m \mathfrak{a})$. Notice, that this step does not work like that for the *usual* relative group $E(\Phi, R, s^h t^m \mathfrak{a})$. The reason are the obstinate denominators in the exponent, which force to reiterate the procedure several times, according to the length of the conjugating element.

Lemma 21. *If p, q and k are given, there exist h and m such that*

$${}^{E^1\left(\Phi, \frac{1}{s^k} R\right)} E(\Phi, s^h t^m R, s^h t^m \mathfrak{a}) \subseteq E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}).$$

Proof. Indeed, one has ${}^h\big({}^g x\big) = {}^{(hgh^{-1})h} x$. Thus,

$$\begin{aligned} {}^{E^1\left(\Phi, \frac{R}{s^k}\right)} E(\Phi, s^h t^m R, s^h t^m \mathfrak{a}) = {}^{E^1\left(\Phi, \frac{R}{s^k}\right)} \Big({}^{E(\Phi, s^h t^m R)} E(\Phi, s^h t^m R)\Big) = \\ = {}^{{}^{E^1\left(\Phi, \frac{R}{s^k}\right)} E(\Phi, s^h t^m R)} \Big({}^{E^1\left(\Phi, \frac{R}{s^k}\right)} E(\Phi, s^h t^m R)\Big). \end{aligned}$$

Now, by the preceding lemma, for any given p and q there exist sufficiently large h and m such that the exponent is contained in $E(\Phi, s^p t^q R)$, while the base is contained in $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$. It remains to recall that by the very definition $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$ is normalised by $E(\Phi, s^p t^q R)$. $\square$

Now, since h and m in Lemma 20 do not depend on the ideal $\mathfrak{a}$, the preceding lemma immediately implies the following fact.

Lemma 22. *If p, k are given, then there is an q such that*

$$^{E^1\left(\Phi,\frac{R}{s^k}\right)}\big[E(\Phi, s^q R, s^q \mathfrak{a}), E(\Phi, s^q R, s^q \mathfrak{b})\big] \subseteq \big[E(\Phi, s^p R, s^p \mathfrak{a}), E(\Phi, s^p R, s^p \mathfrak{b})\big].$$

Iterated application of the above lemma, gives the following result.

Lemma 23. *If p, k and L are given, then there is an q such that*

$$^{E^L\left(\Phi,\frac{R}{s^k}\right)}\big[E(\Phi, s^q R, s^q \mathfrak{a}), E(\Phi, s^q R, s^q \mathfrak{b})\big] \subseteq \big[E(\Phi, s^p R, s^p \mathfrak{a}), E(\Phi, s^p R, s^p \mathfrak{b})\big].$$

Now, we are all set for the next round of calculations. Namely, it is our intention to obtain similar formulae, admitting denominators not only in the exponent, but also on the ground level.

8. Relative commutator calculus

To implement second localisation, we will have to be able to fight powers of *two* elements in the denominator. The relative commutator calculus turns out to be much more technically demanding, than the relative cojugation calculus. Not only that the first step of induction is *by far* the hardest one. Actually, even the usually trivial first *substep* of the first step, the case of two non-opposite roots, turns out to be a real challenge. As always, it is extremely important for the sequel that the resulting power estimates do not depend on the ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Throughhought, we continue to assume $\mathrm{rk}(\Phi) \geq 2$. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ we additionally assume that $2 \in R^*$. These standing assumptions will not be repeated in the statements of lemmas.

Lemma 24. *If p, q, k, m are given, then there exist l and n such that*

$$\left[E^1\left(\Phi, \frac{t^l}{s^k}\mathfrak{a}\right), E^1\left(\Phi, \frac{s^n}{t^m}\mathfrak{b}\right)\right] \subseteq \big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big].$$

These l and n depend on Φ, p, q, k, m alone, and do not depend on the choice of ideals $\mathfrak{a}$ and $\mathfrak{b}$.

Proof. Let $\alpha, \beta \in \Phi$, $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. We have to prove that

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right)\right] \in \big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big].$$

The partition into cases is exactly the same as in the proof of Lemma 20, but the calculations themselves — and the resulting length bounds, should we attempt to record them — are now much fancier.

Case 1. Let $\alpha \neq -\beta$. Then using the Chevalley commutator formula we get

$$\left[x_\alpha\left(\frac{t^l}{s^k}a\right), x_\beta\left(\frac{s^n}{t^m}b\right)\right] = \prod_{i,j>0} x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\left(\frac{t^l}{s^k}a\right)^i\left(\frac{s^n}{t^m}b\right)^j\right) =$$

$$= \prod_{i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}\Big(N_{\alpha\beta ij}s^{jn-ik}t^{il-jm}a^i b^j\Big).$$

Clearly, one can take sufficiently large l and n. It suffices to show, that taking large enough l and n we can redistribute powers of s and t between the first and the second parameters in each factor on the right hand side in such a way, that the resulting product can be expressed as a *product* of commutators without denominators.

Warning. This is one of the key new points in the whole argument, where we cannot thoughtlessly imitate [30] or [62]. Namely, expressing an element as a product of commutators without denomitators, with parameters sitting where they should, is not quite the same as just observing that taking large enough l and n we can kill all the denominators in each factor on the right hand side of the Chevalley commutator formula. This is precisely the point, where the cases $\Phi = \mathrm{B}_2, \mathrm{G}_2$ require *substantial* extra care.

• First, assume that the right hand side of the Chevalley commutator formula consists of one factor. In this case

$$\Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big), x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big] = x_{\alpha+\beta}(N_{\alpha\beta}s^{n-k}t^{l-m}ab).$$

Taking $n \geq 2p+k$ and $l \geq 2q+m$ we can rewrite this commutator as a commutator without denominators as follows:

$$\Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big), x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big] = \big[x_\alpha(s^pt^qa), x_\beta(s^{n-k-p}t^{l-m-q}b)\big].$$

Observe, that the right hand side belongs to $\big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big]$.

The assumption of this item amounts to saying that $|\alpha| = |\beta|$, with the sole exception of two short roots in G_2, whose sum is a short root, where the right hand side of the Chevalley commutator formula consists of *three* factors, rather than one.

Thus, in fact, we have established somewhat more, than claimed. Namely, assume that if $\gamma = \alpha+\beta$, $|\alpha| = |\beta|$, and, moreover, the mutual angle of α and β is not $2\pi/3$ if α, β are short roots of $\Phi = \mathrm{G}_2$. Then for any $h \geq 2p$, any $r \geq 2q$, any $a \in \mathfrak{a}$ and any $b \in \mathfrak{b}$ one has

$$x_{\alpha+\beta}(N_{\alpha\beta}s^ht^rab) \in \big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big].$$

In particular, this proves Case 1 for *simply* laced systems.

• Actually, with the use of the above argument it is easy to completely settle also the case of *doubly* laced systems, except for $\Phi = \mathrm{C}_2$. Indeed, for doubly laced systems it accounts for the case, where $|\alpha| = |\beta|$. Now, let α and β have distinct lengths. If necessary, replacing $[x, y]$ by $[y, x] = [x, y]^{-1}$, we can assume that α is long, and β is short. In this case

$$y = \Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big), x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big] = x_{\alpha+\beta}(N_{\alpha\beta11}s^{n-k}t^{l-m}ab)x_{\alpha+2\beta}(N_{\alpha\beta12}s^{2n-k}t^{l-2m}ab^2).$$

Now, if $\Phi = \mathrm{F}_4$, every root embeds in a subsystem of type A_2. In other words, the root $\alpha+\beta$ is a sum of two *short* roots, whereas $\alpha+2\beta$ is a sum of two *long* roots. Thus, taking $2n \geq 2p+k$ and $l \geq 2q+2m$, we see that each elementary

unipotent on the right hand side of the above formula is itself a single commutator in $\big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big]$.

The cases B_l, $l \geq 3$ and C_l, $l \geq 3$, are treated in a similar way, and are only marginally trickier.

• First, let $\Phi = \mathrm{B}_l$, $l \geq 3$. In this case every *long* root is a sum of two long roots. Clearly, the first elementary factor in expression of the commutator

$$z = \Big[x_\alpha\Big(s^pt^{l-m-q}a\Big), x_\beta\Big(s^{n-k-p}t^qb\Big)\Big] = x_{\alpha+\beta}(N_{\alpha\beta11}s^{n-k}t^{l-m}ab)\cdot$$
$$x_{\alpha+2\beta}(N_{\alpha\beta12}s^{2n-2k-p}t^{l-m+q}ab^2),$$

coincides with the first elementary factor of the above commutator. If $l \geq 2q + m$ and $n \geq 2p + k$, one has $z \in \big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big]$. On the other hand, if, moreover, $l \geq 2q + 2m$, then the long root unipotent

$$yz^{-1} = x_{\alpha+2\beta}(N_{\alpha\beta12}(s^{2n-k}t^{l-2m} - s^{2n-2k-p}t^{l-m+q})ab^2)$$

is also a single commutator in $\big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big]$, by the first item.

• Now, let $\Phi = \mathrm{C}_l$, $l \geq 3$. In this case every *short* root is a sum of two short roots. Set $p' = p$ if $p \equiv k \pmod 2$, and $p' = p+1$ otherwise. Then *second* elementary factor in expression of the commutator

$$z = \Big[x_\alpha\Big(s^{p'}t^{l-2m-2q}a\Big), x_\beta\Big(s^{(2n-k-p')/2}t^qb\Big)\Big] = x_{\alpha+\beta}(N_{\alpha\beta11}s^{(2n-k+p')/2}t^{l-2m-q}ab)\cdot$$
$$x_{\alpha+2\beta}(N_{\alpha\beta12}s^{2n-k}t^{l-2m}ab^2),$$

coincides with the second elementary factor of the commutator y. If $l \geq 3q+2m$ and $n \geq (2p+k+1)/2$, one has $z \in \big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big]$. On the other hand, if, moreover, $n \geq (5p + k + 1)/2$, then the short root unipotent

$$yz^{-1} = x_{\alpha+\beta}(N_{\alpha\beta11}(s^{n-k}t^{l-m} - s^{(2n-k+p')/2}t^{l-2m-q})ab)$$

is also a single commutator in $\big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big]$, by the first item.

• Finally, let $\Phi = \mathrm{C}_2$ or G_2. We will see that under assumption $2 \in R^*$ the proof is essentially the same, as in the above cases. First, let $\Phi = \mathrm{C}_2$, and let α, β, $\alpha \neq \pm\beta$, be two short roots. Then by the first item one has

$$x_{\alpha+\beta}(2s^ht^rab) \in \big[E(\Phi, s^pt^qR, s^pt^q\mathfrak{a}), E(\Phi, s^pt^qR, s^pt^q\mathfrak{b})\big].$$

Since $2 \in R^*$, it follows that $x_{\alpha+\beta}(2s^ht^rab)$ is a single commutator of requested shape, whenever $h \geq 2p$ and $r \geq 2q$. Now, we can repeat exactly the same argument, as in the case $\Phi = \mathrm{B}_l$, $l \geq 3$.

Next, let $\Phi = \mathrm{G}_2$. First, observe that by the first item $x_\alpha(s^ht^rab)$ is already a single commutator of the required shape for any $h \geq 2p$ and any $r \geq 2q$. Now, let

α, β be two short roots, whose sum is a short root. Then the Chevalley commutator formula takes the following form

$$[x_\alpha(\xi), x_\beta(\zeta)] = x_{\alpha+\beta}(\pm 2\xi\zeta)x_{2\alpha+\beta}(\pm 3\xi^2\zeta)x_{\alpha+2\beta}(\pm 3\xi\zeta^2),$$

see, for example, [59, 19] or [86].

Now, setting here $\xi = s^p t^{r-q} a$ and $\zeta = s^{h-p} t^q b$, for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$, we see that

$$x_{\alpha+\beta}(\pm 2 s^h t^r ab) x_{2\alpha+\beta}(\pm 3 s^{h+p} t^{2r-q} a^2 b) x_{\alpha+2\beta}(\pm 3 s^{2h-p} t^{r+q} ab^2) \in \big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big],$$

for any $h \geq 2p$ and $r \geq 2q$. Since each of the resulting long root elements is already a single commutator of requested shape, and $2 \in R^*$, one sees that $x_{\alpha+\beta}(s^h t^r ab)$ is a product of at most three commutators of requested shape. Now, we conclude the analysis of this case by exactly the same argument, as in the case of $\Phi = \mathrm{G}_2$, and conclude that for any two linearly independent roots, any $\alpha \in \mathfrak{a}$, $b \in \mathfrak{b}$, and any $n \geq 2p + 3k$, $l \geq 2q + 3m$, one has

$$\left[x_\alpha\Big(\frac{t^l}{s^k} a\Big), x_\beta\Big(\frac{s^n}{t^m} b\Big)\right] \in \big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big],$$

in fact, the commutator on the left hand side is the product of not more than eight commutators of two elementary unipotents, belonging to $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$ and $E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})$, respectively.

Case 2. Let $\alpha = -\beta$, and one of the following holds, α is short or $\Phi \neq \mathrm{C}_l$. By Lemma 1, there are roots γ and δ such that $\gamma + \delta = \alpha$ and $N_{\gamma\delta 11} = 1$. We can assume that k and l are even and decompose $x_\alpha\Big(\dfrac{t^l}{s^k} a\Big)$ as follows

$$x_\alpha\Big(\frac{t^l}{s^k} a\Big) = \left[x_\gamma\Big(\frac{t^{l/2}}{s^{k/2}}\Big), x_\delta\Big(\frac{t^{l/2}}{s^{k/2}} a\Big)\right] \prod_{i\gamma + j\delta \in \Phi} x_{i\gamma + j\delta}\Big(-N_{\gamma\delta ij} \frac{t^{l(i+j)/2}}{s^{k(i+j)/2}} a^j\Big), \qquad (1)$$

where the product on the right hand side is taken over all roots $i\gamma + j\delta \neq \alpha$. Consider the commutator formula

$$\Big[[y,z] \prod_{i=1}^t u_i, x\Big] = \prod_{i=1}^t {}^{[y,z]\prod_{j=1}^{i-1} u_j}\big[u_i, x\big]\big[[y,z], x\big] \qquad (2)$$

where by convention $\prod_{j=1}^0 u_j = 1$. Now let $y = x_\gamma\Big(\dfrac{t^{l/2}}{s^{k/2}}\Big)$, $z = x_\delta\Big(\dfrac{t^{l/2}}{s^{k/2}} a\Big)$ and u_i's stand for the terms $x_{i\gamma+j\delta}(*)$ in Equation 1. Let $x = x_\beta\Big(\dfrac{s^n}{t^m} b\Big)$ and plug these in to Equation 2. The terms $[u_i, x]$ are all of the form considered in Case 1, and thus for suitable l and n they belong to $\big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big]$. Thus, it immediately follows that $\prod_{i=1}^t {}^{[y,z]\prod_{j=1}^{i-1} u_j}\big[u_i, x\big]$ belongs to this commutator group.

We are left to show that for a suitable q

$$[[y,z],x]=\Big[\Big[x_\gamma\Big(\frac{t^{l/2}}{s^{k/2}}\Big),x_\delta\Big(\frac{t^{l/2}}{s^{k/2}}a\Big)\Big],x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big]$$

is also in $\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big]$. Consider the conjugate

$$\begin{aligned}{}^{x_\gamma\left(-\frac{t^{l/2}}{s^{k/2}}\right)}\Big[\Big[x_\gamma\Big(\frac{t^{l/2}}{s^{k/2}}\Big),x_\delta\Big(\frac{t^{l/2}}{s^{k/2}}a\Big)\Big],x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big]&=\\ {}^{x_\gamma\left(-\frac{t^{l/2}}{s^{k/2}}\right)}\Big[\Big[x_\delta\Big(\frac{t^{l/2}}{s^{k/2}}a\Big),x_\gamma\Big(\frac{t^{l/2}}{s^{k/2}}\Big)\Big]^{-1},x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big].\end{aligned}$$

By the Hall—Witt identity it can be rewritten as

$$\begin{aligned}uv={}&{}^{x_\delta\left(\frac{t^{l/2}}{s^{k/2}}a\right)}\Big[x_\gamma\Big(-\frac{t^{l/2}}{s^{k/2}}\Big),\Big[x_\beta\Big(\frac{s^n}{t^m}b\Big),x_\delta\Big(-\frac{t^{l/2}}{s^{k/2}}a\Big)\Big]^{-1}\Big]\cdot\\ &{}^{x_\beta\left(\frac{s^n}{t^m}b\right)}\Big[x_\delta\Big(\frac{t^{l/2}}{s^{k/2}}a\Big),\Big[x_\gamma\Big(-\frac{t^{l/2}}{s^{k/2}}\Big),x_\beta\Big(-\frac{s^n}{t^m}b\Big)\Big]^{-1}\Big].\end{aligned}$$

Let us consider the factors separately.

Since $\gamma,\delta\neq-\beta$, by Case 1 one can find suitable l and n such that the commutator $\Big[x_\beta\Big(\frac{s^n}{t^m}b\Big),x_\delta\Big(-\frac{t^{l/2}}{s^{k/2}}a\Big)\Big]$ belongs to $\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big]$, and it immediately follows that u belongs to this group.

Applying Chevalley commutator formula to the internal commutator in v, we have

$$v={}^{x_\beta\left(\frac{s^n}{t^m}b\right)}\Big[x_\delta\Big(\frac{t^{l/2}}{s^{k/2}}a\Big),\prod_{i\gamma+j\beta\in\Phi}x_{i\gamma+j\beta}\Big(-N_{\gamma\beta ij}\Big(-\frac{t^{l/2}}{s^{k/2}}\Big)^i\Big(-\frac{s^n}{t^m}b\Big)^j\Big)\Big].$$

Now, for suitable l and n all $x_{i\gamma+j\beta}\Big(-N_{\gamma\beta ij}\Big(-\frac{t^{l/2}}{s^{k/2}}\Big)^i\Big(-\frac{s^n}{t^m}b\Big)^j\Big)$ belong to $E(\Phi,s^{p'}t^{q'}\mathfrak{b})$ for any prescribed p' and q'. Now employing Lemma 20 twice, we can secure that for suitable l and n the second factor v also belongs to the commutator group $\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big]$, and we are done.

Case 3. Let $\Phi=\mathrm{C}_l$ and $\alpha=-\beta$ be a long root. Let $\Phi=\mathrm{C}_l$ and $\alpha=-\beta$ be a long root. By Lemma 1 there exist roots γ and δ such that either $\gamma+2\delta=\beta$ and $N_{\gamma\delta12}=1$, or $2\gamma+\delta=\beta$ and $N_{\gamma\delta21}=1$. Like in the proof of Lemma 20, we lose nothing by looking at the *second* case. Increasing k and l, in necessary, we can assume that k and l are divisible by 3 and decompose $x_\alpha\Big(\frac{t^l}{s^k}a\Big)$ as follows

$$x_\alpha\Big(\frac{t^l}{s^k}a\Big)=\Big[x_\gamma\Big(\frac{t^{l/3}}{s^{k/3}}\Big),x_\delta\Big(\frac{t^{l/3}}{s^{k/3}}a\Big)\Big]\prod_{i\gamma+j\delta\in\Phi}x_{i\gamma+j\delta}\Big(-N_{\gamma\delta ij}\frac{t^{l(i+j)/3}}{s^{k(i+j)/3}}a^j\Big),\qquad(3)$$

where the product is taken over all $(i,j)\neq(1,2)$. Now, repeating the same argument as in Case 2, one can find suitable l and n such that

$$\Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big),x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big]\in\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big],$$

as claimed. ☐

Lemma 25. *If p,q,k,m and L are given, there exist l and n, independent of L, such that*

$$\Big[E^L\Big(\Phi,\frac{t^l}{s^k}\mathfrak{a}\Big),E^1\Big(\Phi,\frac{s^n}{t^m}\mathfrak{b}\Big)\Big]\subseteq\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big].$$

Proof. An easy induction, using identity (C2), shows that

$$\Big[\prod_{i=1}^K u_i,x\Big]=\prod_{i=1}^K {}^{\prod_{j=1}^{K-i}u_j}\big[u_{K-i+1},x\big],$$

where by convention $\prod_{j=1}^0 u_j=1$. This, with the fact that $E(\Phi,s^pt^qR,s^pt^q\mathfrak{a})$ and $E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})$ are both normalized by $E(\Phi,s^Pt^QR)$, where $P\geq p$, $Q\geq q$, show that this lemma immediately follows from the previous one. ☐

Recall that $E\Big(\Phi,\frac{t^l}{s^k}R,\frac{t^l}{s^k}\mathfrak{a}\Big)$ is generated by all elements of the form ${}^ux_\alpha\Big(\frac{t^l}{s^k}a\Big)$, where $u\in E^L\Big(\Phi,\frac{t^l}{s^k}R\Big)$, for some L, and $a\in\mathfrak{a}$.

Lemma 26. *If p,q,k,m are given, there exist l and n such that*

$$\Big[E\Big(\Phi,\frac{t^l}{s^k}R,\frac{t^l}{s^k}\mathfrak{a}\Big),E^1\Big(\Phi,\frac{s^n}{t^m}\mathfrak{b}\Big)\Big]\subseteq\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big].$$

Proof. Obviously, it suffices to prove that for any given p,q,k,m and any L, there exist l and n independent of L such that

$$\Big[{}^{E^L\left(\Phi,\frac{t^l}{s^k}R\right)}E^1\Big(\Phi,\frac{t^l}{s^k}\mathfrak{a}\Big),E^1\Big(\Phi,\frac{s^n}{t^m}\mathfrak{b}\Big)\Big]\subseteq\big[E(\Phi,s^pt^qR,s^pt^q\mathfrak{a}),E(\Phi,s^pt^qR,s^pt^q\mathfrak{b})\big],\tag{4}$$

after that the lemma follows from (4) and identity (C2).

Let $x\in E^L\Big(\Phi,\frac{t^l}{s^k}R\Big)$, $y\in E^1\Big(\Phi,\frac{t^l}{s^k}\mathfrak{a}\Big)$ and $z\in E^1\Big(\Phi,\frac{s^n}{t^m}\mathfrak{b}\Big)$. Using (C2) and the Hall—Witt identity we can write

$$\begin{aligned}[{}^xy,z]=\big[y[y^{-1},x],z\big]={}^y\big[[y^{-1},x],z\big]\cdot[y,z]={}^{yx^{-1}}\Big({}^x\big[[y^{-1},x],z\big]\Big)\cdot[y,z]=\\ {}^{yx^{-1}}\Big({}^y\big[x^{-1},[z,y]\big]\cdot{}^{z^{-1}}\big[y^{-1},[x^{-1},z^{-1}]\big]\Big)\cdot[y,z].\end{aligned}$$

Now Lemma 24, along with the fact that $E(\Phi, s^p t^q R, s^p t^q \mathfrak{a})$ and $E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})$ are both normal in $E(\Phi, s^p t^q R)$, imply that for suitable l and n all three commutators $[y, z]$, ${}^y\big[x^{-1}, [z, y]\big]$ and $\big[y^{-1}, [x^{-1}, z^{-1}]\big]$ are in

$$\big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big].$$

Now, we can invoke Lemma 22 to ensure that there are suitable l and n such that the conjugate ${}^{z^{-1}}\big[y^{-1}, [x^{-1}, z^{-1}]\big]$, and therefore the whole commutator $[{}^x y, z]$, is in $\big[E(\Phi, s^p t^q R, s^p t^q \mathfrak{a}), E(\Phi, s^p t^q R, s^p t^q \mathfrak{b})\big]$. $\square$

9. Mixed commutator formula: localisation proof

Now we are all set to complete a localisation proof of Theorem 1. In fact, we will prove a much more powerful result, in the spirit of Theorem 5.3 of [30]. We start with the following lemma, whose proof mimics the proof of [30], Lemma 5.2, modulo replacing elementary factors by commutators, and correcting some misprints.

Lemma 27. *Fix an element $s \in R$, $s \neq 0$. Then for any k and p there exists an r such that for any $a \in \mathfrak{a}$, any $g \in G(\Phi, R, s^r \mathfrak{b})$ and any maximal ideal $\mathfrak{m}$ of R, there exists an element $t \in R \backslash \mathfrak{m}$, and an integer l such that*

$$\left[x_\alpha\Big(\frac{t^l}{s^k}a\Big), F_s(g)\right] \in \Big[E\big(\Phi, F_s(s^p R), F_s(s^p \mathfrak{a})\big), E\big(\Phi, F_s(s^p R), F_s(s^p \mathfrak{b})\big)\Big]. \qquad (5)$$

Note that here q will depend on the choice of x_α.

Proof. By 5.1 one has $G(\Phi, R) = \varinjlim G(\Phi, R_i)$, where the limit is taken over all finitely generated subrings of R. Thus, without loss of generality we may assume that R is Noetherian. To be specific, we can replace R by the ring generated by a, s and the matrix entries of g in a faithful polynomial representation.

Since $R_\mathfrak{m}$ is a local ring, by Lemma 12 we have the decomposition

$$G(\Phi, R_\mathfrak{m}, \mathfrak{b}_\mathfrak{m}) = E(\Phi, R_\mathfrak{m}, \mathfrak{b}_\mathfrak{m}) T(\Phi, R_\mathfrak{m}, \mathfrak{b}_\mathfrak{m}).$$

Thus, one can decompose $F_\mathfrak{m}(g)$ as $F_\mathfrak{m}(g) = uh$ where $u \in E(\Phi, R_\mathfrak{m}, \mathfrak{b}_\mathfrak{m}) \leq G(\Phi, R_\mathfrak{m})$ and $h \in T(\Phi, R_\mathfrak{m}, \mathfrak{b}_\mathfrak{m})$.

Since $G(\Phi, R_M) = \varinjlim G(\Phi, R_t)$, over all $t \in R \setminus M$, and the same holds for $E(\Phi, s^q R_M)$, $T(\Phi, R_M, s^q R_M)$, etc., we can find an element $t \in R \setminus M$ such that already $F_t(g)$ can be factored as $F_t(g) = uh$, where $u \in E(\Phi, R_t, s^q R_t)$ and $z \in T(\Phi, R_t, s^q R_t)$.

On the other hand, since R is assumed to be Noetherian, R_s is also Noetherian and by Lemma 13 there exists an n such that the canonical homomorphism

$$F_t : G(\Phi, R_s, t^n R_s) \longrightarrow G(\Phi, R_{st})$$

is injective. Next, we take any $l > n$. Since $x_\alpha\Big(\dfrac{t^l}{s^k}a\Big) \in G(\Phi, R_s, t^n\mathfrak{a}_s)$, and the principal congruence subgroup $G(\Phi, R_s, t^n\mathfrak{a}_s)$ is normal in $G(\Phi, R_s)$, one has

$$x = \Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big), F_s(g)\Big] \in G(\Phi, R_s, t^n\mathfrak{a}_s) \le G(\Phi, R_s, t^n R_s).$$

Consider the image $F_t(x) \in G(\Phi, R_{st})$ of x under localisation with respect to t. Since F_t is a homomorphism, one has

$$F_t(x) = \Big[F_t\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_{st}(g)\Big].$$

Now $F_{st}(g)$ can be factored as $F_{st}(g) = F_s(u)F_s(h) \in G(\Phi, R_{st})$. It follows that

$$\begin{aligned} F_t(x) \quad &= \quad \Big[F_t\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_s(u)F_s(h)\Big] = \\ & \Big[F_t\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_t(u)\Big] \cdot {}^{F_t(u)}\Big[F_s\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_s(h)\Big]. \end{aligned}$$

Now, for all cases apart from the case, where $G(\Phi, R) = G_{\mathrm{ad}}(\mathrm{C}_l, R)$, and α is a long root, by Lemmas 2 or 3 one can choose a decomposition $F_t(g) = uh$, where h commutes with $x_\alpha(*)$. Therefore,

$$F_t(x) = \Big[F_t\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_t(u)\Big].$$

Now, by Lemma 26 one can choose such l and n that

$$F_t(x) \in \big[E(\Phi, F_{st}(s^p t^q R), F_{st}(s^p t^q \mathfrak{a})), E(\Phi, F_{st}(s^p t^q R), F_{st}(s^p t^q \mathfrak{b}))\big], \tag{6}$$

considered as a subgroup of $G(\Phi, R_{st})$. In general, this is the first factor of the above expression for $F_t(x)$.

In the exceptional case we can choose $h = h_{\varpi_l}(\varepsilon)$, for some $\varepsilon \equiv 1 \pmod{s^r\mathfrak{b}}$. Clearly, also $\varepsilon^{-1} \equiv 1 \pmod{s^r\mathfrak{b}}$, and thus

$$\Big[F_s\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_s(h)\Big] = x_\alpha\big(t^l s^{r-k} F_{st}(ab)\big),$$

for some $b \in \mathfrak{b}$. Now a reference to Lemmas 17 and 21 shows that one can choose such l and n that the second factor

$${}^{F_t(u)}\Big[F_s\Big(x_\alpha\Big(\frac{t^l}{s^k}a\Big)\Big), F_s(h)\Big] = {}^{F_t(u)}x_\alpha\big(t^l s^{r-k} F_{st}(ab)\big)$$

sits in the same commutator subgroup, as the first factor. Thus, in all cases we get inclusion (6). In other words, $F_t(x)$ can be expressed as

$$F_t(x) = \prod_{i=1}^{L} \Big[{}^{x_{-\beta_i}(F_{st}(s^p t^q c_i))}x_{\beta_i}\big(F_{st}(s^p t^q a_i)\big), {}^{x_{-\gamma_i}(F_{st}(s^p t^q d_i))}x_{\gamma_i}\big(F_{st}\big(s^p t^q b_i\big)\big)\Big],$$

for some $\beta_i, \gamma_i \in \Phi$, some $a_i \in \mathfrak{a}$, $b_i \in \mathfrak{b}$ and some $c_i, d_i \in R$.

Form the following product of commutators in $G(\Phi, R_s)$,

$$y = \prod_{i=1}^{L} \Big[{}^{x_{-\beta_i}(F_s(s^p t^q c_i))} x_{b_i a}\big(F_s(s^p t^q a_i)\big), {}^{x_{-\gamma_i}(F_s(s^p t^q d_i))} x_{\gamma_i}\big(F_s\big(s^p t^q b_i\big)\big) \Big],$$

by the very construction, $F_t(x) = F_y(y)$. On the other hand, $x, y \in G(\Phi, R_s, t^n R_s)$ and the restriction of F_t to $G(\Phi, R_s, t^n R_s)$ is injective by Lemma 13, it follows $x = y$ and thus we established (5). $\square$

Now we are in a position to finish the proof of Theorem 1 and, in fact, of the following much stronger result. Morally, it culminates all calculations of Sections 7, 8 and 9, and asserts that for any elements $g_1, \dots, g_K \in E(\Phi, R_s, \mathfrak{a}_s)$, in finite number, and any s-adic neighborhoods Y and Z of e in the *elementary* subgroups $E(\Phi, R, \mathfrak{a})$ and $E(\Phi, R, \mathfrak{b})$, respectively, there exists a small s-adic neighbourhood X of e in the principal congruence subgroup $G(\Phi, R, \mathfrak{b})$ such that $[g_i, F_s(X)] \subseteq F_s([Y, Z])$, for all i. This is a very powerful result, which will be used in this form in the proposed description of some classes of intermediate subgroups. See [84, 63] for a clarification, why one needs commutator formulae in this stronger form. In turn, this result can be easily deduced from Lemma 27 by a standard patching argument using partitions of 1.

Theorem 2. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi) \geq 2$*. In the cases $\Phi = \mathrm{C}_2, \mathrm{G}_2$ assume additionally that $2 \in R^*$. Then for any $s \in R$, $s \neq 0$, any p, k and L, there exists an r such that for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a commutative ring R, one has*

$$\Big[E^L\Big(\Phi, \frac{1}{s^k}R, \frac{1}{s^k}\mathfrak{a}\Big), F_s\big(G(\Phi, R, s^r\mathfrak{b})\big)\Big] \subseteq \Big[E\big(\Phi, F_s(s^p R), F_s(s^p\mathfrak{a})\big), E\big(\Phi, F_s(s^p R), F_s(s^p\mathfrak{b})\big)\Big]. \quad (7)$$

Proof. First we claim that for the same k and L and *any* q there exists an r such that

$$\Big[E^1\Big(\Phi, \frac{1}{s^k}\mathfrak{a}\Big), F_s\big(G(\Phi, R, s^r\mathfrak{b})\big)\Big] \subseteq \Big[E\big(\Phi, F_s(s^q R), F_s(s^q\mathfrak{a})\big), E\big(\Phi, F_s(s^q R), F_s(s^q\mathfrak{b})\big)\Big]. \quad (8)$$

Indeed, let $x_\alpha\Big(\frac{1}{s^k}a\Big) \in E^1\Big(\Phi, \frac{1}{s^k}\mathfrak{a}\Big)$, and $g \in G(\Phi, R, s^r\mathfrak{b})$. For any maximal ideal $\mathfrak{m} \lhd R$, choose an $t_\mathfrak{m} \in R\backslash\mathfrak{m}$ and a positive integer $l_\mathfrak{m}$ according to (5). Since the collection of all $t_\mathfrak{m}^{l_\mathfrak{m}}$ is not contained in any maximal ideal, we may find a finite number of them, $t_1^{l_1}, \dots, t_K^{l_K}$ and such $c_1, \dots, c_K \in R$ that

$$t_1^{l_1}c_1 + \dots + t_K^{l_K}c_K = 1.$$

It follows that

$$x_\alpha\Big(\frac{1}{s^k}a\Big) = x_\alpha\Big(a\sum_{i=1}^K \frac{t_i^{l_i}}{s^k}c_i\Big) = \prod_{i=1}^K x_\alpha\Big(\frac{t_i^{l_i}}{s^k}c_i a\Big).$$

Since there are only finitely many factors, it follows from (5) that for *any* h there exists an r such that

$$\Big[x_\alpha\Big(\frac{t_i^{l_i}}{s^k}c_i a\Big), F_s(g)\Big] \in \Big[E\big(\Phi, F_s(s^h R), F_s(s^h\mathfrak{a})\big), E\big(\Phi, F_s(s^h R), F_s(s^h\mathfrak{b})\big)\Big]. \qquad (9)$$

A direct computation using (9), Formula (C2) and Lemma 23, shows that if h was large enough, we get

$$\Big[x_\alpha\Big(\frac{1}{s^k}a\Big), F_s(g)\Big] = \Big[\prod_{i=1}^K x_\alpha\Big(\frac{t_i^{l_i}}{s^k}c_i a\Big), F_s(g)\Big] \in \\ \Big[E\big(\Phi, F_s(s^q R), F_s(s^q\mathfrak{a})\big), E\big(\Phi, F_s(s^q R), F_s(s^q\mathfrak{b})\big)\Big].$$

This proves our claim.

Now, applying to (8) the commutator formula (C2), we see that if q was large enough, we get

$$\Big[E^L\Big(\Phi, \frac{1}{s^k}R, \frac{1}{s^k}\mathfrak{a}\Big), F_s\big(G(\Phi, R, s^r\mathfrak{b})\big)\Big] \subseteq \\ {}^{E^{L-1}\left(\Phi, \frac{1}{s^k}R, \frac{1}{s^k}\mathfrak{a}\right)}\Big[E\big(\Phi, F_s(s^p R), F_s(s^p\mathfrak{a})\big), E\big(\Phi, F_s(s^p R), F_s(s^p\mathfrak{b})\big)\Big].$$

To finish the proof it only remains to once more invoke Lemma 23. $\square$

First proof of Theorem 1. To get the inclusion of the left hand side into the right hand side, set $s = 1$ in Theorem 2. Inclusion in the other direction is obvious. $\square$

10. Relative versus absolute, and variations

Using the absolute standard commutator formula and calculations of Sections 4 and 6 we can give another proof of Theorem 1 — but not of the stronger Theorem 2.

Second proof of Theorem 1. By Lemma 5 one has

$$\big[E(\Phi, R, \mathfrak{a}), G(\Phi, R, \mathfrak{b})\big] = \Big[\big[E(\Phi, R), E(\Phi, R, \mathfrak{a})\big], G(\Phi, R, \mathfrak{b})\Big].$$

Since all subgroups here are normal in $G(\Phi, R)$, Lemma 5 implies

$$\big[E(\Phi, R, \mathfrak{a}), G(\Phi, R, \mathfrak{b})\big] \le \\ \le \Big[E(\Phi, R, \mathfrak{a}), \big[E(\Phi, R), G(\Phi, R, \mathfrak{b})\big]\Big] \cdot \Big[E(\Phi, R), \big[E(\Phi, R, \mathfrak{a}), G(\Phi, R, \mathfrak{b})\big]\Big].$$

Applying the *absolute* standard commutator formula [27, Theorem 1] = Lemma 10 above, to the first factor on the right hand side, we immediately see that it *coincides* with $[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})]$.

On the other hand, applying to the second factor on the right hand Lemma 19 followed by Lemma 10, we can conclude that it is *contained* in

$$\big[E(\Phi,R),G(\Phi,R,\mathfrak{ab})\big]=E(\Phi,R,\mathfrak{ab})\leq\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})\big].$$

Thus, the left hand side is contained in the right hand side, the inverse inclusion being obvious. $\square$

Lemma 5 asserts that the commutator of two elementary subgroups, one of which is absolute, is itself an elementary subgroup. One can ask, whether one always has

$$\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})\big]=E(\Phi,R,\mathfrak{ab})?$$

Easy examples show that in general this equality may fail quite spectacularly. In fact, when $\mathfrak{a}=\mathfrak{b}$, one can only conclude that

$$E(\Phi,R,\mathfrak{a}^2)\leq\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{a})\big]\leq E(\Phi,R,\mathfrak{a}).$$

with right bound attained for some *proper* ideals, such as an ideal $\mathfrak{a}$ generated by an idempotent.

Nevertheless, the true reason, why the equality in Lemma 5 holds, is not the fact that one of the ideals $\mathfrak{a}$ or $\mathfrak{b}$ coincides with R, but only the fact that $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal.

Theorem 3. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi)\geq 2$. *When $\Phi=\mathrm{B}_2$ or $\Phi=\mathrm{G}_2$, assume moreover that R has no residue fields $\mathbb{F}_2$ of 2 elements. Further, let R be a commutative ring and $\mathfrak{a},\mathfrak{b}\trianglelefteq R$ be two comaximal ideals of R, i.e., $\mathfrak{a}+\mathfrak{b}=R$. Then one has the following equality*

$$\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})\big]=E(\Phi,R,\mathfrak{ab}).$$

Proof. First of all, observe that by Lemmas 5 and 16 one has

$$E(\Phi,R,\mathfrak{a})=\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R)\big]=\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{a})\cdot E(\Phi,R,\mathfrak{b})\big].$$

Thus,

$$\begin{aligned}E(\Phi,R,\mathfrak{a})\leq\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{a})\big]\cdot\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})\big]\leq\\ \leq\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{a})\big]\cdot E(\Phi,R,\mathfrak{ab}).\end{aligned}$$

Commuting this inclusion with $E(\Phi,R,\mathfrak{b})$, we see that

$$\begin{aligned}\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})\big]\leq\Big[\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{a})\big],E(\Phi,R,\mathfrak{b})\Big]\cdot\\ \big[E(\Phi,R,\mathfrak{ab}),E(\Phi,R,\mathfrak{b})\big].\end{aligned}$$

The absolute standard commutator formula, applied to the second factor, shows that its is contained in

$$\big[G(\Phi,R,\mathfrak{ab}),E(\Phi,R,\mathfrak{b})\big]\le\big[G(\Phi,R,\mathfrak{ab}),E(\Phi,R)\big]=E(\Phi,R,\mathfrak{ab}).$$

On the other hand, applying to the first factor Lemma (C3), and then again the absolute standard commutator formula, we see that it is contained in

$$\begin{aligned}\Big[\big[E(\Phi,R,\mathfrak{a}),E(\Phi,R,\mathfrak{b})\big],E(\Phi,R,\mathfrak{a})\Big]&\le\\ &\le\big[G(\Phi,R,\mathfrak{ab}),E(\Phi,R,\mathfrak{a})\big]\le\\ &\qquad\le\big[G(\Phi,R,\mathfrak{ab}),E(\Phi,R)\big]=E(\Phi,R,\mathfrak{ab}).\end{aligned}$$

Together with Lemma 16 this finishes the proof. □

11. Where next?

In this section we state and very briefly discuss some further relativisation problems, related to the results of the present paper. We are convinced that these problems can be successfully addressed with our methods. Throughout we assume that $\mathrm{rk}(\Phi)\ge 2$.

Outside of some initial observations in Sections 3, 4, and 6, in the present paper we consider only the usual relative subgroups depending on one ideal of the ground ring, rather than relative subgroups defined in terms of admissible pairs. In fact, calculations necessary to unwind the relative commutator calculus are already awkward enough with one parameter, especially in rank 2. After some thought, we decided not to overcharge the first exposition of our method in this setting with unwieldy technical details. Actually, most of these details are immaterial for the method itself. This suggests the following problems.

Problem 1. Develop working versions of relative conjugation calculus and relative commutator calculus, for relative subgroups corresponding to admissible pairs.

Problem 2. Prove the relative standard commutator formula

$$\big[E(\Phi,R,\mathfrak{a},\mathfrak{c}),C(\Phi,R,\mathfrak{b},\mathfrak{d})\big]=\big[E(\Phi,R,\mathfrak{a},\mathfrak{c}),E(\Phi,R,\mathfrak{b},\mathfrak{d})\big].$$

There is little doubt that what one needs to solve these problems is a stubborn combination of the methods of the present paper with those developed by Michael Stein in [57]. Solution of the following problem is also in sight, and would require mostly technical efforts.

Problem 3. Obtain explicit length estimates in the relative conjugation calculus and relative commutator calculus.

Let us mention some further problems, where we hope to apply methods of the present paper. Firstly, we have in mind description of subnormal subgroups of Chevalley groups.

Problem 4. Describe subnormal subgroups of a Chevalley group $G(\Phi, R)$.

It is well known that this problem is essentially a special case of the following more general problem.

Problem 5. Describe subgroups of a Chevalley group $G(\Phi, R)$, normalised by the relative elementary subgroup $E(\Phi, R, \mathfrak{q})$, for an ideal $\mathfrak{q} \trianglelefteq R$.

Conjectural answer may be stated as follows: there exists an integer $m = m(\Phi)$, depending only on Φ, with the following property. For any subgroup $H \leq G(\Phi, R)$ normalised by $E(\Phi, R, \mathfrak{q})$ there exist an ideal $\mathfrak{a} \trianglelefteq R$ such that

$$E(\Phi, R, \mathfrak{q}^m\mathfrak{a}) \leq H \leq C(\Phi, R, \mathfrak{a}).$$

The ideal $\mathfrak{a}$ is unique up to equivalence relation $\Diamond_{\mathfrak{q}}$.

The real challenge is to find the smallest possible value of m. For instance, for the case of $\mathrm{GL}(n, R)$, $n \geq 3$, it has taken the following values:

- $m = 7$ for $n \geq 4$, John Wilson, 1972 [90],
- $m = 24$ (under some stability conditions), Anthony Bak, 1982 [6],
- $m = 6$, Leonid Vaserstein, 1986 [70],
- $m = 48$, Li Fuan and Liu Mulan, 1987 [38],
- $m = 5$, the second author 1990 [74],
- $m = 4$, Vaserstein 1990 [72].

An exposition of these results with detailed proofs may be found in [95]. Clearly, [6] and [38] drop out of the mainstream. The reason is that [6] was published some 15 years after completion, and [38] relied upon [6]. Nevertheless, these papers are very pertinent in what concerns discussion of equivalence relation $\Diamond_{\mathfrak{q}}$.

For other classical groups the best known results are due to Gerhard Habdank [22, 23] and the third author [95]–[97], under assumption $2 \in R^*$, and to You Hong, in general, see the discussion in [32].

For exceptional groups there are no published results. Recently, the second and the third authors have modified the third generation proof of the main structure theorems [80, 81], and obtained the following values: $m = 7$ for Chevalley groups of types E_6 and E_7. This result will be published in a separate paper. But to get results with the same bound for groups of type E_8 one will have to use localisation.

Other problems we intend to address with relative concern description of various classes of intermediate subgroups, see [76, 88, 39] for a survey. In [63] we specifically discuss how localisation comes into play. Let us mention two of the most immediate such problems.

Problem 6. Describe the following classes of subgroups

- subgroups in $\mathrm{GL}(27, R)$, containing $E(\mathrm{E}_6, R)$,
- subgroups in $\mathrm{Sp}(56, R)$, containing $E(\mathrm{E}_7, R)$.

These problems are discussed by Alexander Luzgarev in [40], where one can find conjectural answers. Before that the second author and Victor Petrov [83, 84, 85, 49, 51], and independently and simultaneously You Hong [91, 92, 93] decribed overgroups of classical groups, in the corresponding $\mathrm{GL}(n, R)$. The proofs of these results partly relied on localisation. Immediately thereafter Alexander Luzgarev described subgroups of $G(\mathrm{E}_6, R)$, containing $E(\mathrm{F}_4, R)$, in his splendid paper [41], also using localisation, see also [42].

Also, we propose to apply the methods of the present paper to describe overgroups of subsystem subgroups in exceptional groups.

Problem 7. Describe subgroups in $G(\Phi, R)$, containing $E(\Delta, R)$, under assumption that $\Delta^{\perp} = \emptyset$ and all irreducible components of Δ except maybe one have rank ≥ 2.

The following problem appeared as Problem 9 in [28]. It seems to be extremely challenging, and would certainly require the full force of localisation-completion. Its solution would be a simultaneous generalisation of the results in [30, 9], as also of our Theorem 1.

Problem 8. Let R be a ring of finite Bass—Serre dimension $\delta(R) = d < \infty$, and let (I_i, Γ_i), $1 \leq i \leq m$, be form ideals of (R, Λ). Prove that for any $m > d$ one has

$$\Big[\big[\ldots[G(\Phi, R, I_1), G(\Phi, R, I_2)], \ldots\big], G(\Phi, R, I_m)\Big] = \Big[\big[\ldots[E(\Phi, R, I_1), E(\Phi, R, I_2)], \ldots\big], E(\Phi, R, I_m)\Big].$$

Let us also reiterate very ambitious Problems 7 and 8 posed in [32]. The first of these problems refers to the context of odd unitary groups, as created by Victor Petrov [49, 50, 51].

Problem 9. Generalise results of the present paper to odd unitary groups.

One of the first steps towards a solution of this problem, and other related problems for odd unitary groups was recently done by Rabeya Basu [15].

The next problem refers to the recent context of isotropic reductive groups. Of course, it only makes sense over commutative rings, but on the other hand, a lot of new complications occur, due to the fact that relative roots do not form a root system, and the interrelations of the elementary subgroup with the group itself are abstruse even over fields (the Kneser—Tits problem). Still, we are convinced that after the recent breakthrough by Victor Petrov and Anastasia Stavrova [52, 55] most necessary tools are already there. See also their subsequent papers with Alexander Luzgarev and Ekaterina Kulikova [43, 35].

Problem 10. Obtain results similar to those of the present paper for [groups of points of] isotropic reductive groups.

Of course, here one shall have to develop the whole conjugation and commutator calculus almost from scratch.

Results of the present paper were first announced in our joint paper [28] with Alexei Stepanov. We thank him for numerous extremely useful discussions.

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